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Strong Tits alternatives for compact 3-manifolds with boundary

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Abstract

We show that the fundamental group of a compact orientable irreducible 3-manifold with boundary is either virtually abelian or has a finite index subgroup that surjects onto a non-abelian free group. We also show that the fundamental group of a compact orientable 3-manifold with boundary whose components are not all spheres is either virtually abelian or is SQ-universal.

MSC: 20E40; 57N10

1. Introduction

Various concepts of what it means for an infinite group to be "large" have appeared in the literature, with the papers [17,4] introducing precision into this notion. The idea is to describe a group theoretic property P as "large" if: a pre-image of a group with P also has P; finite index subgroups and supergroups of groups with P also possess P and (in [4] but not in [17]) a quotient of a group with P by a finite group also has P. (However we do not insist that arbitrary supergroups of groups with P also have P.) A standard example of such a property is whether a group G contains a non-abelian free group and by Tits' classic result [21] we know that if k is a field of characteristic 0 then any finitely generated subgroup (in fact any subgroup) of $GL(n,k)$ is either virtually soluble or contains a non-abelian free group, and these options are mutually exclusive. A whole host of other results have appeared since then along these lines, so that now one says that the Tits alternative holds for a given class of groups if every group in that class has one of these two properties.

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However there are stronger measures of "largeness": one is that of a group G being SQ-universal, namely every countable group is a subgroup of a quotient of G (note that finite index subgroups and supergroups of SQ-universal groups are SQ-universal by [13]). Yet another "large" property is that G has a finite index subgroup H which surjects onto a non-abelian free group, and this is stronger still because a non-abelian free group is SQ-universal. To see these three notions are not equivalent, even in the class of finitely presented groups, an infinite finitely presented simple group cannot be SQ-universal because of Higman's famous result that there are uncountably many non-isomorphic finitely generated groups, so we can use Scott's example [19] of such a group containing the free group F_2 of rank 2. As for the other case, it was shown in [11, Theorem V.10.3] that all non-trivial free products with the exception of $Z_2 * Z_2$ are SQ-universal, but in [17] the exact criterion for a free product $G_1 * G_2$ to have a finite index subgroup surjecting onto (without loss of generality) F_2 was obtained. It is that either one of the factors has that property, or that the factors each have proper subgroups of finite index which are not both of order 2. Therefore as infinite simple groups have no proper subgroups of finite index, a free product of an infinite finitely presented simple group with any non-trivial finitely presented soluble group will give a finitely presented example.

In [17] the idea of the "large" property $P(G)$ generated by a group G is defined. This is then used to obtain the notion of another group H being larger than G, namely that H has $P(G)$. As a consequence, if P is any "large" property possessed by G than H will have that property too. Under this definition F_2 is larger than any finitely generated group, so that if P is a "large" property enjoyed by a single finitely generated group then all finitely generated groups that are equally as large as F_2, that is which possess $P(F_2)$, will also possess P. But the property $P(F_2)$ has in fact been mentioned earlier: namely that of having a finite index subgroup which surjects onto F_2. Thus for finitely generated groups this can be thought of as the strongest notion of "largeness", and consequently we can remove the quotation marks from now on and unambiguously define a finitely generated group to be large if it has a finite index subgroup surjecting onto F_2, which is in line with usage in recent papers.

So given a class $\mathscr{C}$ of finitely generated groups that is known or believed to satisfy the Tits alternative, it is reasonable to make the following definition:

Definition 1. We say that $\mathscr{C}$ satisfies the strong Tits alternative if every group in $\mathscr{C}$ is virtually soluble or is large. We also say that $\mathscr{C}$ satisfies the semi-strong Tits alternative if every group in $\mathscr{C}$ is virtually soluble or is SQ-universal.

This fits with the terminology of [14], where it is shown that subgroups of finitely generated Coxeter groups satisfy the strong Tits alternative. In this paper we are interested in the class of fundamental groups of compact 3-manifolds, which will be finitely presented. For fundamental groups of closed 3-manifolds the Tits alternative is unknown as yet (see [5] Theorem 2.9 and [16] for partial results), however it is implied by Thurston's geometrisation conjecture (or more specifically the infinite fundamental group part: this states that if M is closed, orientable and irreducible with $\pi_1 M$ infinite but not containing a subgroup of the form $\mathbb{Z}+\mathbb{Z}$ then M is a closed hyperbolic

3-manifold). This can be seen by decomposing M into prime factors (we can assume M is oriented). It is enough to show that each of these factors P_i has a fundamental group that is either virtually soluble or contains a non-abelian free group. This is the case if $\pi_1 P_i$ is finite, and if it is infinite, not equal to $\mathbb{Z}$, and does not contain $\mathbb{Z}+\mathbb{Z}$ then the conjecture implies that P_i is a closed hyperbolic 3-manifold, so $\pi_1 P_i$ contains a non-abelian free group. If $\pi_1 P_i$ contains $\mathbb{Z}+\mathbb{Z}$ then P_i is irreducible and so we can use Proposition 2.8 of [20] to conclude that P_i is either Haken or a Siefert fibre space. In the first case [6] Corollary 4.10 gives that $\pi_1 P_i$ is soluble or contains a non-abelian free group, and if P_i is a Siefert fibre space then we can quotient out by the infinite cyclic normal subgroup to get a surjection onto a two-dimensional orbifold group F. This will contain a non-abelian free group unless F is elementary, in which case it is virtually soluble, so $\pi_1 P_i$ satisfies this alternative too.

We also have the class of fundamental groups of compact 3-manifolds with boundary, where we assume that the boundary components are not all spheres (or spheres and projective planes in the non-orientable case) because filling in a sphere boundary component leaves the fundamental group unchanged, so if we did not have this condition then all fundamental groups of closed 3-manifolds would be included. In this case more is known, for instance the Tits alternative holds for this class of groups as shown in [5, Theorem 2.9]. Also Theorem 5 of [18] shows using the Euler characteristic of cell complexes that if G is the fundamental group of a compact 3-manifold and G has negative Euler characteristic then G is SQ-universal; the Euler characteristic condition on G is shown to be exactly that G is infinite but is not the fundamental group of a compact aspherical orientable 3-manifold which is closed or which has boundary consisting solely of tori, nor the fundamental group of a non-orientable 3-manifold whose orientable double cover is such a 3-manifold. Then more recently there is the result of [3] that if M is a compact orientable irreducible 3-manifold with non-empty incompressible boundary then either the boundary of M consists only of tori and is covered by the product of the torus times the interval, or the fundamental group of M is large. The proof comes out of methods which look for essential surfaces in 3-manifolds.

In this paper we strengthen these results. We show that if M is a compact 3-manifold with non-empty boundary and with $\pi_2(M)=0$ then the strong Tits alternative holds for $\pi_1 M$, and $\pi_1 M$ is large unless it is one of only four groups. We then show that the fundamental group of any compact 3-manifold M with non-empty boundary whose components are not all spheres and projective planes satisfies the semi-strong Tits alternative. As for the strong Tits alternative for this class of groups, progress is hindered by our lack of knowledge of whether these alternatives hold for closed 3-manifolds, because we can form a connected sum of any closed 3-manifold with any compact 3-manifold with boundary which gives rise to a compact 3-manifold with boundary. However we isolate the necessary property in showing that the strong Tits alternative will be satisfied for all fundamental groups of compact 3-manifolds with non-empty boundary whose components are not all spheres and projective planes if and only if every non-simply connected closed 3-manifold has a non-trivial finite cover. We also show that the semi-strong Tits alternative holds for all discrete Möbius groups and finish by giving examples of a closed orientable non-simply connected 4-manifold with no non-trivial finite covers, and a closed orientable 4-manifold whose fundamental group is not virtually soluble but does not contain a non-abelian free group.

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2. Alternatives for compact orientable irreducible 3-manifolds

Given a finitely presented group G, the deficiency of a finite presentation for G is the number of generators minus the number of relators. The deficiency $d(G)$ of G is the supremum over all finite presentations for G; by abelianising it is seen that $d(G) \leqslant \beta_1(G)$ which is the number of infinite cyclic summands in the abelianisation G/G'. We have two very useful theorems connecting deficiency with large groups:

Theorem 2 (Baumslag and Pride [1]). *If $d(G) \geqslant 2$ then G is large.*

Theorem 3 (Howie [9, Theorem A]). *Suppose that $\bar{K}$ is a connected regular covering complex of a finite 2-complex K, with non-trivial free abelian covering transformation group A. Suppose also that $H_2(\bar{K}, F)$ has a free FA-submodule of rank at least $1 + \chi(K)$ for some field F, where $\chi(K)$ is the Euler characteristic of K. Then $\pi_1(K)$ is large.*

We can apply these theorems to $G = \pi_1(M)$ for M a compact orientable irreducible 3-manifold with non-empty boundary as follows: irreducibility implies that no boundary components can be spheres (unless M is the 3-ball which is simply connected) and that $\pi_2(M) = 0$ by the sphere theorem. Now G is infinite because the boundary of M contains a surface of non-negative Euler characteristic and the universal cover $\tilde{M}$ has $\pi_n\tilde{M} = 0$ for all n, so that M is aspherical. This means that given any finite aspherical CW-complex K with $\pi_1(K) = G$ we have that K is homotopy equivalent to M by the Whitehead Theorems (as $\pi_i(K) = \pi_i(M)$ for all i) and so the homology groups $H_i(K) = H_i(M) = H_i(G)$.

Now as M has boundary we can certainly find an aspherical CW-complex K of no more than 2 dimensions having one 0-cell and with $\pi_1(K) = G$. Then K gives rise to a presentation of G consisting of n generators (the 1-cells) and r relators (the 2-cells) so that

$$d(G) \geqslant n - r = 1 - \chi(K) = 1 - \chi(M) = 1 - \frac{\chi(\partial M)}{2}.$$

But ∂M contains no spheres so that $\chi(\partial M) \leqslant 0$ with equality if and only if ∂M consists of only tori. Thus this gives us that $d(G) \geqslant 1$ and $d(G) \geqslant 2$ if the boundary contains a surface of negative Euler characteristic. In fact, Theorem 4 of [18] shows that $d(G)$ is equal to $1 - \chi(\partial M)/2$, and in that paper Theorem 2 above is then immediately applied to conclude that G is SQ-universal (and in fact large) when $\chi(\partial M) < 0$. Our intention is to use Howie's strengthening of this result to deal with the case where $\chi(\partial M) = 0$.

We use Corollary 2.4 in [9] which states that if G has a finite presentation of deficiency 1 and has a normal subgroup N such that G/N is free abelian of rank $n > 0$ then, choosing a finite 2-complex K with $\pi_1(K) = G$ and $\chi(K) = 0$ and a covering complex $\bar{K}$ corresponding to N, we have that G is large if there is some field F with $H_2(\bar{K},F) \neq 0$. This is because $H_2(\bar{K},F)$ contains a non-zero free submodule, as $F[G/N]$ is a domain, so that we can apply Theorem 3. This gives us:

Theorem 4. *If G is the fundamental group of a compact orientable irreducible* 3-*manifold M with boundary consisting of $k \geqslant 3$ tori then G is large.*

Proof. We have $\beta_1(G) = \operatorname{rk} H_1(M,\mathbb{Q}) \geqslant k \geqslant 3$, which can be shown by a standard Mayer–Vietoris argument: we sew a solid torus into each boundary component to create a closed orientable 3-manifold $\bar{M}$, and start at the $H_2(\bar{M})$ term of the exact sequence which has the same rank as $H_1(\bar{M})$ by Poincaré-Lefschetz duality. Thus taking a boundary component T of M with $\pi_1 T = \langle x,y\rangle$, we let C be the normal closure of x and y in G. Note that as M is irreducible all torus boundary components must be incompressible because otherwise the loop theorem implies that there is a simple closed curve on T which bounds a disc in the interior of M. But then the boundary of a small neighbourhood of the disc unioned with T is a sphere (along with T) which must bound a ball, so we have a solid torus. If $\theta : G \to \bar{G}$ is the natural map onto the abelianisation of G then as $\bar{G}$ has rank at least 3 we can find $H \leqslant \bar{G}$ with $\theta(x), \theta(y) \in H$ and $\bar{G}/H = \mathbb{Z}$, so that $N = \theta^{-1}(H)$ contains C and $G/N = \mathbb{Z}$. Now N corresponds to a regular infinite cyclic cover M' of M, so let π be the covering map. Although $N = \pi_1 M'$ may be infinitely generated, we still have $\pi_i M' = 0$ for all $i \geqslant 2$. We know that G has a deficiency 1 presentation and that we can choose a finite 2-complex K with $\pi_1 K = G$ such that K is aspherical (for instance a cell decomposition of M which is itself aspherical), so that the covering $\bar{K}$ of K corresponding to N is also aspherical and hence $\bar{K}$ and M' are homotopy equivalent, with $H_2(M',\mathbb{Q}) = H_2(\bar{K},\mathbb{Q})$.

We now show that $H_2(M',\mathbb{Q}) \neq 0$. We know M' is not compact and the boundary of M' (which is a union of surfaces without boundary) must cover that of M. Picking a basepoint p in the torus boundary component T of M and a basepoint q in M' projecting down to p, we have that π restricted to the connected component of $\partial M'$ containing q is a covering map of T but as N contains x and y this covering must correspond to $\pi_1(T)$ and so is a homeomorphism. Thus M' has a closed surface as one of its boundary components which is a torus. However if we now look at the Mayer-Vietoris sequence obtained by sewing in a solid torus S into this boundary component of M', certainly $H_3(M' \cup S) = 0$ (taking coefficients over $\mathbb{Q}$) as $M' \cup S$ is a 3-manifold that is not closed (because there exists boundary of M that is not in T) and $H_2(M' \cap S) = \mathbb{Q}$, so if $H_2(M') = 0$ we lose exactness as $H_2(M' \cap S)$ cannot inject into $H_2(M') \oplus H_2(S) = 0$. □

We can now extend this result to other compact orientable irreducible 3-manifolds M with boundary consisting of only tori. Clearly not every such manifold has a fundamental group that is large; for instance, the solid torus and the torus times the interval. However if M has a finite cover with at least 3 boundary components (which will all be tori also) then $\pi_1 M$ will be large. We now follow through and adapt the proof of Lemma 2.1 in [3] which assumed that the interior of M has a complete hyperbolic structure with finite volume to obtain.

Theorem 5. *Let M be a compact orientable irreducible 3-manifold with non-empty boundary, all components of which are tori. Then either $\pi_1 M$ has a finite cover with at least 3 boundary components or M is a Siefert fibre space.*

Proof. Taking a torus T in ∂M with $i : T \to M$ the inclusion map and $m \in A = i_*(\pi_1 T)$ of infinite order, we ask: does m commute with x^2 for all $x \in \pi_1 M$? If not then we take x such that $y = [x^2, m]$ is non-trivial. As M is compact, orientable and irreducible with boundary we have that $H_1(M, \mathbb{Z})$ is infinite and so M contains a properly embedded 2-sided incompressible surface, that is M is Haken. By a theorem of Hempel [8] $\pi_1 M$ is residually finite. This means that there is a finite group F and a surjective homomorphism θ of $\pi_1 M$ onto F with $\theta(y)$ non-trivial. The finite covering corresponding to the kernel of θ has at least 3 torus boundary components covering T because the index of $\theta(A)$ in F equals the number of components of the pre-image of T in the covering space. But if the index of $\theta(A)$ in F were less than 3 then $\theta(x^2) \in \theta(A)$ which is abelian, so $\theta(y) = \theta([x^2, m])$ would be trivial which is not true.

Otherwise the centraliser C of m contains every square in $\pi_1 M$, hence it also contains the subgroup S of $\pi_1 M$ generated by all the squares. But S is normal and $\pi_1 M/S$ is a finitely generated group of exponent 2 so it is abelian and therefore finite, meaning that M has a finite cover N whose fundamental group C has the element m of infinite order in its centre. Thus N is a compact orientable 3-manifold with infinite fundamental group and is irreducible (because its universal cover is irreducible, by Hempel [7, Theorem 13.4] using the fact that M is Haken, or by Meeks et al. [12] which merely requires that M is irreducible) with $\pi_1 N$ having an infinite cyclic normal subgroup. This means that N is a Siefert fibre space (by Hempel [7, Corollary 12.8] as here N has boundary and so is Haken, which was first proved by Waldhausen although it is now known to be true without the Haken condition). Thus M is also a Siefert fibre space (for instance by using the criterion in [10, Definition 1.36] that for compact 3-manifolds with infinite fundamental group, Siefert fibre spaces are those which are finitely covered by circle bundles over surfaces). $\square$

We now obtain the strong Tits alternative for the fundamental groups of compact orientable irreducible 3-manifolds with boundary.

Corollary 6. *If M is a compact orientable irreducible 3-manifold with non-empty boundary then either $\pi_1 M$ is large or M is the 3-ball, the solid torus, the torus times the interval or the twisted I-bundle over the Klein bottle, with $\pi_1 M$ trivial, $\mathbb{Z}$, $\mathbb{Z} + \mathbb{Z}$ or the fundamental group of the Klein bottle, respectively.*

Proof. We know that $\pi_1 M$ is large if $\chi(\partial M) < 0$ by Theorem 2, or if $\chi(\partial M) = 0$ and M is not a Siefert fibre space by Theorems 4 and 5. If M is a Siefert fibre space then $G = \pi_1 M$ has an infinite normal cyclic subgroup Z with $G/Z = F$ for F the group of a 2 dimensional orbifold, and as $\partial M \neq \emptyset$ we get that F must be a free product of cyclic groups (see [7, p. 118]) and thus F, and hence G, is large unless F is cyclic or $Z_2 * Z_2$. If we are in the latter case then F and consequently G would be soluble so, using Theorem 4.2 of [6], we have that M is as above. $\square$

We can now deal with non-orientable 3-manifolds M with boundary. We say that M is P^2-irreducible if it is irreducible and in addition contains no embedded 2-sided projective planes.

Corollary 7. *If M is a compact non-orientable P^2-irreducible 3-manifold with non-empty boundary then either $\pi_1 M$ is large or M is the solid Klein bottle, the twisted I-bundle over the torus or the Klein bottle times the interval, with $\pi_1 M$ equal to $\mathbb{Z}$, $\mathbb{Z}+\mathbb{Z}$ or the fundamental group of the Klein bottle, respectively.*

Proof. As M is P^2-irreducible it has no boundary components which are spheres and no projective planes in the boundary either as they would be 2-sided. We can assume that $\pi_1 M$ is infinite (as otherwise it would be equal to $\mathbb{Z}_2$ but then M would have projective planes in the boundary; see [7, Theorem 9.5]). Then the orientable double cover M' is compact with non-empty boundary and is irreducible (for instance one could use Lemma 10.4 of [7] which states that any double cover of M is P^2-irreducible). Applying Corollary 6 to M' gives us that $\pi_1 M'$ is either large or soluble, so the same holds for $\pi_1 M$. In the soluble case, we again look at the list in [6, Theorem 4.2]. □

Note that our alternatives above for compact 3-manifolds with non-empty boundary which are orientable and irreducible or non-orientable and P^2-irreducible extend at once to any compact 3-manifold M with boundary and with $\pi_2(M)=0$. This is because M must be a connected sum of such a manifold with a homotopy 3-sphere, so its fundamental group is unchanged.

3. Strong alternatives for compact 3-manifolds with boundary

Having dealt with irreducible 3-manifolds in the previous section, we can now look more generally at compact 3-manifolds with boundary.

Proposition 8. *If M is a compact 3-manifold with non-empty boundary whose components are not all spheres or projective planes then the semi-strong Tits alternative holds for $\pi_1 M$.*

Proof. We can assume M is orientable (as it is enough to prove the conclusion for the orientable double cover) which will have non-empty boundary not consisting entirely of spheres. We first fill in any boundary components that are spheres and then we decompose M into a finite connected sum of prime orientable factors (see [7, Theorem 3.15]). At least one of these factors must have some boundary, one component of which will be a closed orientable surface of non-positive Euler characteristic. This factor N is compact, orientable and irreducible (with infinite fundamental group) and so Corollary 6 applies to $\pi_1 N$. If $\pi_1 N$ is large then as $\pi_1 M$ naturally surjects onto $\pi_1 N$, we conclude that $\pi_1 M$ is large and hence is SQ-universal. If $\pi_1 N$ is virtually abelian then either some other prime factor of M has non-trivial fundamental group so that $\pi_1 M$ is a non-trivial free product not equal to $\mathbb{Z}_2 * \mathbb{Z}_2$ and so is SQ-universal by Lyndon and Schupp [11, Theorem V.10.3], or all other factors are trivial so that $\pi_1 M = \pi_1 N$. $\square$

It might be wondered why we cannot conclude that $\pi_1 M$ satisfies the strong Tits alternative. It all comes down to whether there exists a closed 3-manifold with infinite fundamental group having no non-trivial finite covering.

Theorem 9. *If $\mathscr{F}$ is the class of fundamental groups of compact 3-manifolds with non-empty boundary whose components are not all spheres and projective planes then the strong Tits alternative holds for $\mathscr{F}$ if and only if every closed 3-manifold with infinite fundamental group has a non-trivial finite covering.*

Proof. If we have a strange closed 3-manifold S with no non-trivial finite covers then the connected sum of S with, say, the solid torus has boundary a torus and fundamental group $\pi_1 S * \mathbb{Z}$. This is SQ-universal but not large by Pride's result [17], because $\pi_1 S$ has no non-trivial finite index subgroups and $\pi_1 S$ is not large (if $\pi_1 S$ were large than it would have to surject onto F_2 and hence onto $\mathbb{Z}_n$).

If however there are no such closed 3-manifolds then we can follow the proof of Proposition 8 for any compact orientable 3-manifold with non-empty boundary whose components are not all spheres (and for the fundamental groups in $\mathscr{F}$ of non-orientable 3-manifolds we again use the orientable double cover). We take a prime factor N with boundary and if $\pi_1 N$ is soluble then it certainly has subgroups of finite index greater than 2. Then either all other prime factors are simply connected so that $\pi_1 M$ is soluble or there exists another factor C with non-trivial fundamental group, which will have a proper subgroup of finite index (as if it has boundary then $\pi_1 C$ will have positive first Betti number and hence subgroups of any finite index, and if C is closed we use the assumption if $\pi_1 C$ is infinite and the trivial subgroup if $\pi_1 C$ is finite). Thus $\pi_1 C * \pi_1 N$ is large and so hence so is $\pi_1 M$. $\square$

Of course, the existence of closed 3-manifolds with no non-trivial finite covers contradicts Thurston's geometrisation conjecture; for instance this implies that all compact 3-manifolds have residually finite fundamental group. At the time of writing, G. Perelman has claimed a proof of the geometrisation conjecture but this has yet to be confirmed. However there is the result of Ol'shanskiĭ [15] that the semi-strong Tits alternative is satisfied by all word-hyperbolic groups; this includes all fundamental groups of closed hyperbolic n-manifolds. This result allows us to extend [18, Corollary 9].

Corollary 10. *Finitely generated discrete subgroups of $PSL(2,\mathbb{C})$ satisfy the semi-strong Tits alternative.*

Proof. We take a torsion free finite index subgroup Γ so that $\mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold M with $\pi_1 M$ finitely generated. If M is not closed then we use Scott's compact core theorem [7, Theorem 8.6] to get a compact 3-submanifold Q of M with $\pi_1(Q) = \pi_1(M)$, and we can assume that Q has no sphere boundary components because

M is irreducible, so they would bound 3-balls in M which we can add to Q. Then Q is not closed so we can use Proposition 8. □

So we ask:

Question 11. *If M is a closed orientable irreducible 3-manifold with $\pi_1 M$ infinite but not containing a $\mathbb{Z}+\mathbb{Z}$ subgroup then is $\pi_1 M$ large?*

An affirmative answer would have very strong consequences: it would imply that there exist finite covers of M with arbitrarily large first Betti number (by taking a finite cover N with $\theta : \pi_1 N \to F_n$ surjective and then considering $\theta^{-1}(F_m)$ for F_m of finite index in F_n), hence with positive first Betti number, hence M is virtually Haken, hence M is hyperbolic. Alternatively the truth of (the infinite fundamental group part of) the geometrisation conjecture reduces our question to whether fundamental groups of closed hyperbolic 3-manifolds are large.

We finish by a construction that demonstrates just how different things are in the world of 4-manifolds.

Proposition 12. *There exists a closed orientable 4-manifold M with infinite fundamental group that has no non-trivial finite covers. There also exists a closed orientable 4-manifold whose fundamental group is not virtually soluble nor contains a non-abelian free group.*

Proof. Given any finitely presented group G, we can form a closed 4-manifold M with $\pi_1 M = G$ by taking M to be the boundary of a regular neighbourhood of a two dimensional CW complex K with $\pi_1 K = G$ embedded in $\mathbb{R}^5$. We first choose G to be the Higman–Thompson group (called V in [2]), or indeed any infinite finitely presented simple group, so that if G had a proper subgroup of finite index we could take one that was normal, and as M has no double cover it is orientable. Then we take G to be Thompson's group (called F in [2]) which is infinite, finitely presented and has the property that every non-abelian subgroup contains a free abelian subgroup of infinite rank. Therefore G and its subgroups contain no non-abelian free group. It is also known that its commutator subgroup G' is infinite and simple with $G/G' = \mathbb{Z}+\mathbb{Z}$. If G were virtually soluble then taking a subgroup H of finite index (which we can assume is soluble and normal in G) would give us $H \cap G'$ is normal in G', so either G' is contained in H but that implies that G' is soluble, or $H \cap G'$ is trivial. However H has finite index in G so $H \cap G'$ has finite index in G'. Finally if the manifold created is non-orientable then the fundamental group of its orientable double cover will still not be virtually soluble, nor contain a non-abelian free group. □

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